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The Hilbert space $L^2(SU(2))$ as a representation space for the group $(SU(2)\times SU(2))$ Ⓢ S_2

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Abstract. The Hilbert space $L^2(SU(2))$ is used as a representation space for a (unitary) representation of the semi-direct product group $(SU(2)\times SU(2))$ Ⓢ S_2 and the corresponding group algebra. Special operators are constructed which are closely related to the representation theory of the groups SU(2) and S_2 and are irreducible tensor operators with respect to $(SU(2)\times SU(2))$ Ⓢ S_2. These operators are then used to define a complete set of irreducible tensor operators and to calculate two classes of Clebsch–Gordan coefficients of $(SU(2)\times SU(2))$ Ⓢ S_2.

1. Introduction

It is well known that the theory of irreducible tensor operators (IT) constitutes an important practical calculus for problems in nearly all fields of theoretical physics. However, to exploit the general results of this theory for a concrete problem one has to know the unitary irreducible representations (unirreps) and Clebsch–Gordan coefficients (CG coefficients) for the group in question and has to define explicitly all the necessary operators in the given Hilbert space. This paper deals mainly with the problem of how to construct by means of 'induction' out of the IT already known with respect to $SU(2)\times SU(2)$, IT with respect to the semi-direct product group $(SU(2)\times SU(2))$ Ⓢ S_2. We use these IT to calculate CG coefficients for the semi-direct product group which are important for the theory of selection rules for physical systems which are (approximate) invariant with respect to this semi-direct product group. Therefore the results of this paper can be applied to all problems where the semi-direct product group appears as (approximate) symmetry group and where the carrier space is isomorphic to (invariant subspaces or coset spaces of) $L^2(SU(2))$. Examples are the (Wick rotated) Bethe–Salpeter equation (Guth and Soper 1975, Böhm *et al* 1973) and the hydrogen atom, where the subspace of bound states is, in a natural way, isomorphic to $L^2(SU(2))$ since there exists a one-to-one mapping between the eigenstates of the hydrogen atom Hamiltonian (in parabolic coordinates) and the elements $D^{j*}_{mm'}$ of $L^2(SU(2))$, and where the semi-direct product group plays the role of a dynamical invariance group (Dirl and Kasperkovitz 1976). If the (function and/or operator) labelling problem is solved, practical calculations can be simplified extremely (expansions of interaction operators in terms of convenient IT). For instance, in the first example both the 'orbital' and 'spin' part of the Bethe–Salpeter wavefunctions transform according to unirreps of the semi-direct product group and the physically relevant states have to be constructed by using CG coefficients of this group.

In the previous paper (Dirl 1976, to be referred to as I), we have investigated problems concerning the construction of convenient operator bases whose elements are IT with respect to SU(2)×SU(2). There we succeeded in defining complete sets of IT composed of special operators which are closely related to the representation theory of SU(2). Besides this we found useful operator identities between these special IT which are of interest for practical calculations.

The conventions and notation of I are used throughout this paper. The material is organized as follows. In § 2 we summarize the properties of the unirreps of (SU(2)× SU(2)) Ⓢ S_2. Besides this, we consider three types of subductions and calculate the corresponding multiplicities. The multiplicities occurring in the reduction of the Kronecker products are also determined quite generally. In § 3 we define a unitary representation of (SU(2)×SU(2)) Ⓢ S_2 and a representation of the corresponding group algebra. In § 4 the 'global' and 'local' definition of an IT with respect to (SU(2)×SU(2)) Ⓢ S_2 is stated; the second one is more useful in defining such IT. The construction of an IT with respect to (SU(2)×SU(2)) Ⓢ S_2 can be simplified extremely if one knows IT with respect to the normal subgroup. IT within the group algebra and special IT closely related to the matrix elements of the unirreps of the group SU(2) are discussed in §§ 5 and 6. These IT allow us to calculate quite generally two classes of CG coefficients for the semi-direct product group. Just as in §§ 5 and 6 we construct in § 7 (without knowing explicitly the CG coefficients of the semi-direct product group) much more general IT composed of IT introduced in I. Thereby a complete set of IT with respect to (SU(2)×SU(2)) Ⓢ S_2 is defined which is equivalent to the set of IT introduced in I.

2. Properties of the unirreps of (SU(2)×SU(2)) Ⓢ S_2

Since we are interested in defining (in $L^2(SU(2))$) IT with respect to (SU(2)× SU(2)) Ⓢ S_2 we need the matrix elements of the unirreps of this group or its Lie algebra. First of all we summarize the definition of this semi-direct product group ($\omega_0=(0,0,0)$):

$$(SU(2)\times SU(2)) ⓢ S_2=\{(\omega_1,\omega_2|r):\omega_i\in SU(2), r(=e,s)\in S_2\} \tag{2.1}$$

$$(\omega_1,\omega_2|e)(\omega_1',\omega_2'|e)=(\omega_1\omega_1',\omega_2\omega_2'|e) \tag{2.2}$$

$$(\omega_0,\omega_0|s)(\omega_1,\omega_2|e)(\omega_0,\omega_0|s)^{-1}=(\omega_2,\omega_1|e) \tag{2.3}$$

$$(\omega_0,\omega_0|r)(\omega_0,\omega_0|r')=(\omega_0,\omega_0|rr'). \tag{2.4}$$

Equations (2.1–4) show that this group is (in a purely algebraic sense) indeed a semi-direct product group with normal subgroup SU(2)×SU(2) (for topological questions see Dirl and Kasperkovitz 1976).

For the unirreps of (SU(2)×SU(2)) Ⓢ S_2 there exists two different types (Dirl and Kasperkovitz 1976),

$$\{D^{jj\kappa}(\omega_1,\omega_2|r):j=0,\tfrac{1}{2},1,\ldots,\kappa=0,1\} \tag{2.5}$$

$$\{D^{j_1j_2}(\omega_1,\omega_2|r):j_1<j_2,j_i=0,\tfrac{1}{2},1,\ldots\}, \tag{2.6}$$

whose matrix elements are given by

$$D^{jj\kappa}_{m_1m_2,m'_1m'_2}(\omega_1,\omega_2|r)=D^{j}_{m'_1,m'_{r1}}(\omega_1)D^{j}_{m_2,m'_{r2}}(\omega_2)(-1)^{\kappa r} \tag{2.7}$$

$$D^{j_1j_2}_{r'm_1m_2,r''m'_1m'_2}(\omega_1,\omega_2|r)=D^{j_1}_{m_1m'_1}(\omega_{r'1})D^{j_2}_{m_2m'_2}(\omega_{r'2})\delta_{r'r,r''} \tag{2.8}$$

$(m_{ei}=m_i,\ m_{s1}=m_2,\ m_{s2}=m_1,\ \omega_{ei}=\omega_i,\ \omega_{s1}=\omega_2,\ \omega_{s2}=\omega_1)$. Therefore the dimensions of the unirreps (2.5, 6) are

$$\dim D^{jj\kappa}(\omega_1,\omega_2|r)=(2j+1)^2 \tag{2.9}$$

$$\dim D^{j_1j_2}(\omega_1,\omega_2|r)=2(2j_1+1)(2j_2+1). \tag{2.10}$$

It is easily proved by means of equations (2.7,8) and the well known properties of the matrix elements of the unirreps of SU(2) (Rose 1957) that each unirrep of (SU(2)×SU(2)) Ⓢ S_2 is equivalent to its complex conjugate:

$$D^{jj\kappa *}(\omega_1,\omega_2|r)=U^{jj\kappa}D^{jj\kappa}(\omega_1,\omega_2|r)U^{jj\kappa +} \tag{2.11}$$

$$U^{jj\kappa}_{m_1m_2,m'_1m'_2}=(-1)^{2j+m'_1+m'_2}\delta_{m_1,-m'_1}\delta_{m_2,-m'_2,} \tag{2.12}$$

$$D^{j_1j_2 *}(\omega_1,\omega_2|r)=U^{j_1j_2}D^{j_1j_2}(\omega_1,\omega_2|r)U^{j_1j_2+} \tag{2.13}$$

$$U^{j_1j_2}_{r'm_1m_2,r''m'_1m'_2}=(-1)^{j_1+m'_1+j_2+m'_2}\delta_{r',r''}\delta_{m_1,-m'_1}\delta_{m_2,-m'_2}. \tag{2.14}$$

This implies that all primitive characters must be real (Hamermesh 1962) which is of some relevance if one calculates the multiplicities for certain subductions. The characters of the unirreps (2.5,6) are given by

$$\chi^{jj\kappa}(\omega_1,\omega_2|e)=\chi^{j}(\omega_1)\chi^{j}(\omega_2) \tag{2.15}$$

$$\chi^{jj\kappa}(\omega_1,\omega_2|s)=(-1)^{\kappa}\chi^{j}(\omega_1\omega_2) \tag{2.16}$$

$$\chi^{j_1j_2}(\omega_1,\omega_2|r)=\delta_{re}(\chi^{j_1}(\omega_1)\chi^{j_2}(\omega_2)+\chi^{j_2}(\omega_1)\chi^{j_1}(\omega_2)), \tag{2.17}$$

where the quantities $\chi^{j}(\omega_i)=\chi^{j*}(\omega_i)$ denote the primitive characters of SU(2).

By means of the character formula for SU(2) (Hamermesh 1962)

$$m_{jj'j''}=\int \mathrm{d}\mu(\omega)\chi^{j}(\omega)\chi^{j'}(\omega)\chi^{j''}(\omega)=\Delta(jj'j'') \tag{2.18}$$

where $\Delta(jj'j'')$ means the triangle symbol which is completely symmetric in all indices, one can easily calculate the multiplicities for the following subductions:

$$D^{jj\kappa}(\omega_1,\omega_2|r)\downarrow \mathrm{SU(2)\times SU(2)}=D^{jj}(\omega_1,\omega_2) \tag{2.19}$$

$$D^{j_1j_2}(\omega_1,\omega_2|r)\downarrow \mathrm{SU(2)\times SU(2)}=D^{j_1j_2}(\omega_1,\omega_2)\oplus D^{j_2j_1}(\omega_1,\omega_2). \tag{2.20}$$

(The symbol $D^{jj'}(\omega_1,\omega_2)$ introduced in I denotes the unirreps of SU(2)×SU(2).)

$$D^{jj\kappa}(\omega_1,\omega_2|r)\downarrow (\mathrm{SU(2)[\times]SU(2)})\times S_2\simeq\sum_{l=0}^{2j}{}_{\oplus}\, D^{l}(\omega)\oplus D^{\kappa_j(l)}(r),\qquad D^{\kappa_j(l)}(r)=(-1)^{\kappa_j(l)r} \tag{2.21}$$

$$\kappa_j(l)=2j+l+\kappa \qquad \mathrm{mod}\ 2 \tag{2.22}$$

$$D^{j_1j_2}(\omega_1,\omega_2|r)\downarrow (\mathrm{SU(2)[\times]SU(2)})\times S_2\simeq\sum_{j=j_2-j_1;\kappa'=0,1}^{j_1+j_2}{}_{\oplus}\, D^{j}(\omega)\otimes D^{\kappa'}(r). \tag{2.23}$$

(As in I the symbol [×] is used for Kronecker products.)

$$D^{jj\kappa}(\omega_1, \omega_2|r)\downarrow S_2 \simeq \sum_{\kappa'}{}_{\oplus}\, m_{jj\kappa,\kappa'} D^{\kappa'}(r) \tag{2.24}$$

$$m_{jj\kappa,\kappa'} = \tfrac{1}{2}(2j+1)[2j+1+(-1)^{\kappa+\kappa'}] \tag{2.25}$$

$$D^{j_1j_2}(\omega_1, \omega_2|r)\downarrow S_2 \simeq \sum_{\kappa'}{}_{\oplus}\, m_{j_1j_2,\kappa'} D^{\kappa'}(r) \tag{2.26}$$

$$m_{j_1j_2,\kappa'} = (2j_1+1)(2j_2+1). \tag{2.27}$$

The first step which has to be done if one wants to calculate the CG coefficients for (SU(2)×SU(2)) Ⓢ S_2 is to calculate the multiplicities occurring in the Kronecker products:

$$D^{\alpha}(\omega_1, \omega_2|r)\otimes D^{\beta}(\omega_1, \omega_2|r) \simeq \sum_{\gamma}{}_{\oplus}\, m_{\alpha\beta\gamma} D^{\gamma}(\omega_1, \omega_2|r). \tag{2.28}$$

For the sake of brevity we denote the labels $jj\kappa$ or j_1j_2 which distinguish the equivalence classes of the unirreps of (SU(2)×SU(2))Ⓢ S_2 by the symbols $\alpha(\beta, \gamma)$. By using the character formula

$$m_{\alpha\beta\gamma} = \tfrac{1}{2}\sum_{r} \iint d\mu(\omega_1)d\mu(\omega_2)\chi^{\alpha}(\omega_1, \omega_2|r)\chi^{\beta}(\omega_1, \omega_2|r)\chi^{\gamma}(\omega_1, \omega_2|r) \tag{2.29}$$

and equation (2.18), we obtain, through straightforward calculation, the multiplicities

$$m_{jj\kappa,j'j'\kappa',j''j''\kappa''} = \Delta(jj'j'')\delta_{\kappa'',\kappa+\kappa'} \tag{2.30}$$

$$m_{jj\kappa,j'j'\kappa',j_1j_2} = \Delta(jj'j_1)\Delta(jj'j_2) \tag{2.31}$$

$$m_{jj\kappa,j_1'j_2',j_1''j_2''} = \Delta(jj_1'j_1'')\Delta(jj_2'j_2'') + \Delta(jj_1'j_2'')\Delta(jj_2'j_1'') \tag{2.32}$$

$$\begin{aligned} m_{j_1j_2,j_1'j_2',j_1''j_2''} = {} & \Delta(j_1j_1'j_1'')\Delta(j_2j_2'j_2'') + \Delta(j_1j_1'j_2'')\Delta(j_2j_2'j_1'') \\ & + \Delta(j_1j_2'j_1'')\Delta(j_2j_1'j_2'') + \Delta(j_1j_2'j_2'')\Delta(j_2j_1'j_2''). \end{aligned} \tag{2.33}$$

This implies that (SU(2)×SU(2)) Ⓢ S_2 is not a simply reducible group (in the sense of Hamermesh 1962). Only in the cases (2.30,31) are the CG coefficients uniquely determined up to a phase factor.

We denote the CG coefficients of (SU(2)×SU(2))Ⓢ S_2 by

$$\left\{ \begin{Bmatrix} \beta & \beta' & \beta'' \\ p & p' & p'' \end{Bmatrix}_w : \quad \begin{matrix} \beta(\beta', \beta'') = jj\kappa; & p(p', p'') = m_1m_2 \\ \beta(\beta', \beta'') = j_1j_2; & p(p', p'') = r'm_1m_2 \\ w = 1, 2, \ldots, m_{\beta\beta'\beta''} & \end{matrix} \right\}. \tag{2.34}$$

They are elements of unitary matrices, i.e. they must satisfy the usual orthogonality relations. Furthermore they must decompose the reducible representation (2.28) into the desired direct sum of unirreps (CG series):

$$D^{\beta}_{p_1p_2}(\omega_1, \omega_2|r)D^{\beta'}_{p_1'p_2'}(\omega_1, \omega_2|r) = \sum_{\substack{\beta''w \\ p_1''p_2''}} \begin{Bmatrix} \beta & \beta' & \beta'' \\ p_1 & p_1' & p_1'' \end{Bmatrix}_w D^{\beta''}_{p_1''p_2''}(\omega_1, \omega_2|r) \begin{Bmatrix} \beta & \beta' & \beta'' \\ p_2 & p_2' & p_2'' \end{Bmatrix}_w^* \tag{2.35}$$

$$\sum_{\substack{p_1p_1' \\ p_2p_2'}} \begin{Bmatrix} \beta & \beta' & \beta'' \\ p_1 & p_1' & p_1'' \end{Bmatrix}_w^* D^{\beta}_{p_1p_2}(\omega_1, \omega_2|r)D^{\beta'}_{p_1'p_2'}(\omega_1, \omega_2|r) \begin{Bmatrix} \beta & \beta' & \beta'' \\ p_2 & p_2' & p_2'' \end{Bmatrix}_{w'} = D^{\beta''}_{p_1''p_2''}(\omega_1, \omega_2|r)\delta_{ww'} \tag{2.36}$$

3. L^2(SU(2)) as a representation space for (SU(2) × SU(2)) Ⓢ S_2 and the corresponding group algebra

The Hilbert space L^2(SU(2)) which was introduced in I can be used as a representation space for (SU(2) × SU(2)) Ⓢ S_2. The unitary representation U of SU(2)×SU(2) (see equation (2.4) of I) can be extended to a unitary representation V of (SU(2)× SU(2)) Ⓢ S_2 in the following way:

$$V:(\omega_1,\omega_2|r)\to V(\omega_1,\omega_2|r)=U(\omega_1,\omega_2)W(r)$$

$$[V(\omega_1,\omega_2|e)f](\omega)=[U(\omega_1,\omega_2)f](\omega)=f(\omega_1^{-1}\omega\omega_2)$$

$$[V(\omega_0,\omega_0|s)f](\omega)=[W(s)f](\omega)=f(\omega^{-1}). \tag{3.1}$$

However as in I this representation V is not a faithful one. V is only isomorphic to SO(4, R) Ⓢ S_2 where the normal subgroup SO(4, R) is the homomorphic image of SU(2)×SU(2):

$$W(s)U(\omega_1,\omega_2)W^{+}(s)=U(\omega_2,\omega_1). \tag{3.2}$$

As in I we use as basis of L^2(SU(2)) the functions

$$\{Q^j_{mk}: j=0,\tfrac{1}{2},1,\ldots;-j\leq m,k\leq j\}. \tag{I.2.8}$$

By means of the definition (3.1) we obtain

$$U(\omega_1,\omega_2)Q^j_{mk}=\sum_{m'k'}D^{jj}_{m'k',mk}(\omega_1,\omega_2)Q^j_{m'k'} \tag{3.3}$$

$$W(s)Q^j_{mk}=(-1)^{2j}Q^j_{km} \tag{3.4}$$

which show that this basis of L^2(SU(2)) carries a direct sum of the unirreps

$$\{D^{jj\kappa(j)}(\omega_1,\omega_2|r): j=0,\tfrac{1}{2},1,\ldots \text{ with } \kappa(j)=2j \bmod 2\}. \tag{3.5}$$

Each unirrep $D^{jj\kappa(j)}(\omega_1,\omega_2|r)$ occurs only once in V.

Finally one realizes that the subduction (2.21) leads in general to reducible representations of (SU(2)[×]SU(2))×S_2. With the aid of the CG coefficients of SU(2) we obtain immediately the elements $Q^{j;l}_n$ of a new basis of L^2(SU(2)) which transform according to the unirreps $D^l(\omega)\otimes D^{\kappa_j(l)}(r)$ of SU(2)×S_2.

$$Q^{j;l}_n=\sum_k (j\,n-k,jk|ln)Q^j_{n-k,k} \tag{3.6}$$

$$U(\omega,\omega|r)Q^{j;l}_n=(-1)^{lr}\sum_{n'}D^l_{n'n}(\omega)Q^{j;l}_{n'}. \tag{3.7}$$

In analogy to I we introduce by

$$C=\frac{1}{2}\sum_r\iint d\mu(\omega_1)\,d\mu(\omega_2)c(\omega_1,\omega_2|r)V(\omega_1,\omega_2|r),$$

$$c(\omega_1,\omega_2|r)\in L^2((SU(2)\times SU(2))Ⓢ S_2) \tag{3.8}$$

representations of the group algebras ${}^{(i)}\mathscr{A}$((SU(2)×SU(2)) Ⓢ S_2) (i = left or right). For similar reasons to those in § 2 of I, the operators

$$E^{jj\kappa(j)}_{mk,m'k'}=\tfrac{1}{2}[E^{jj}_{mk,m'k'}+(-1)^{\kappa(j)}E^{jj}_{mk,k'm'}W(s)] \tag{3.9}$$

are the only 'units' of ${}^{(i)}\mathscr{A}$((SU(2)×SU(2)) Ⓢ S_2) which do not vanish identically. The operators $E^{jj}_{mk,m'k'}$ appearing in (3.9) are given by equation (2.19) of I.

4. IT with respect to (SU(2) × SU(2)) Ⓢ S_2

The general definition of an IT with respect to (SU(2)×SU(2)) Ⓢ S_2 is given by the transformation properties of its components. An equivalent definition uses the commutation relations of the IT components with the elements of the left- and right-Lie algebras and the transformation properties with respect to the finite group S_2:

$$V(\omega_1, \omega_2|r)T^{JJ\kappa}_{MK}V^+(\omega_1, \omega_2|r) = \sum_{M'K'} D^{JJ\kappa}_{M'K',MK}(\omega_1, \omega_2|r)T^{JJ\kappa}_{M'K'} \tag{4.1}$$

$$\begin{aligned} &[{}^{(L)}J_\pm, T^{JJ\kappa}_{MK}] = [(J\mp M)(J\pm M+1)]^{1/2}T^{JJ\kappa}_{M\pm1,K} \\ &[{}^{(L)}J_3, T^{JJ\kappa}_{MK}] = MT^{JJ\kappa}_{MK} \\ &[{}^{(R)}J_\pm, T^{JJ\kappa}_{MK}] = [(J\mp K)(J\pm K+1)]^{1/2}T^{JJ\kappa}_{M,K\pm1} \\ &[{}^{(R)}J_3, T^{JJ\kappa}_{MK}] = KT^{JJ\kappa}_{MK} \end{aligned} \tag{4.2}$$

$$W(s)T^{JJ\kappa}_{MK}W^+(s) = (-1)^\kappa T^{JJ\kappa}_{KM} \tag{4.3}$$

$$V(\omega_1, \omega_2|r)T^{J_1J_2}_{r'MK}V^+(\omega_1, \omega_2|r) = \sum_{r''M'K'} D^{J_1J_2}_{r''M'K',r'MK}(\omega_1, \omega_2|r)T^{J_1J_2}_{r''M'K'} \tag{4.4}$$

$$\begin{aligned} &[{}^{(L)}J_\pm, T^{J_1J_2}_{eMK}] = [(J_1\mp M)(J_1\pm M+1)]^{1/2}T^{J_1J_2}_{e,M\pm1,K} \\ &[{}^{(L)}J_3, T^{J_1J_2}_{eMK}] = MT^{J_1J_2}_{eMK} \\ &[{}^{(R)}J_\pm, T^{J_1J_2}_{eMK}] = [(J_2\mp K)(J_2\pm K+1)]^{1/2}T^{J_1J_2}_{e,M,K\pm1} \\ &[{}^{(R)}J_3, T^{J_1J_2}_{eMK}] = KT^{J_1J_2}_{eMK} \end{aligned} \tag{4.5}$$

$$W(s)T^{J_1J_2}_{rMK}W^+(s) = T^{J_1J_2}_{sr,M,K}. \tag{4.6}$$

The elements of the left- and right-Lie algebras are given by equations (3.7–10) of I. Because of

$$W(s)^{(L)}J_j = {}^{(R)}J_jW(s), \tag{4.7}$$

which is a consequence of equation (3.2), it is clear that equations (4.5) and (4.6) with $r = e$ suffice to define IT of rank J_1, J_2.

It is obvious that the relations (4.2,3) and (4.5,6) are very useful if one wants to construct IT with respect to (SU(2)×SU(2)) Ⓢ S_2. For if we know IT with respect to SU(2)×SU(2) we are able to 'induce' by means of equation (4.3,6) operators which are IT with respect to (SU(2)×SU(2)) Ⓢ S_2. This is just the 'inverse' of the subduction problem (see equations (2.19,20)). If IT with respect to (SU(2)×SU(2)) Ⓢ S_2 are given and we restrict this group to the normal subgroup then there are two possibilities:

$$T^{JJ\kappa}_{MK} = T^{JJ}_{MK} \tag{4.8}$$

$$T^{J_1J_2}_{eMK} = T^{J_1J_2}_{MK}, \qquad T^{J_1J_2}_{sMK} = T^{J_2J_1}_{KM}. \tag{4.9}$$

(We use for IT with respect to SU(2)×SU(2) the notation introduced in I). Whereas in the first case an IT remains an IT, it always decomposes into two IT in the second case.

To decide which ranks J, J, κ or J_1, J_2 ($J_1 < J_2$) of IT can be realized in $L^2(SU(2))$ it suffices to remember that V is isomorphic to SO(4, R)ⓈS_2 or to investigate the Wigner–Eckart theorem for (SU(2)×SU(2))ⓈS_2. Fortunately we are confronted with the multiplicity-free cases only because of equation (3.5) (for this reason we omit the dummy index w):

$$\langle Q^j_{mk}, T^\beta_p Q^{j'}_{m'k'}\rangle = \left\{\begin{matrix}\beta & j'j'\kappa(j') \\ p & m'k'\end{matrix}\middle|\begin{matrix} jj\kappa(j) \\ mk\end{matrix}\right\}(jj\kappa(j)\|T^\beta\|j'j'\kappa(j')) \tag{4.10}$$

($\beta(p)$ means either $JJ\kappa(MK)$ or $J_1J_2(r'MK)$). It is therefore obvious that the following ranks are allowed:

$$JJ\kappa: \qquad J=0,\tfrac{1}{2},1,\ldots, \qquad \kappa=0,1 \tag{4.11}$$

$$J_1J_2: \qquad J_i=0,1,2,\ldots \quad \text{or} \quad J_i=\tfrac{1}{2},\tfrac{3}{2},\ldots. \tag{4.12}$$

The problem of decomposing a given operator $\mathcal{O}\in V_Q$ in IT components with respect to (SU(2) × SU(2))ⓈS$_2$ can in principle be solved in the following way (Dirl and Kasperkovitz 1976, Dirl 1974a,b):

$$\mathcal{O}=\sum_{\beta p} T^\beta_{pp}[\mathcal{O}] \tag{4.13}$$

$$T^\beta_{pq}[\mathcal{O}]=\tfrac{1}{2}n_\beta\sum_r\iint \mathrm{d}\mu(\omega_1)\,\mathrm{d}\mu(\omega_2)D^{\beta*}_{pq}(\omega_1,\omega_2|r)\,V(\omega_1,\omega_2|r)\mathcal{O}V^+(\omega_1,\omega_2|r) \tag{4.14}$$

$$n_\beta=\begin{cases}(2J+1)^2 & \text{for } \beta=JJ\kappa\\ 2(2J_1+1)(2J_2+1) & \text{for } \beta=J_1J_2(J_1<J_2).\end{cases} \tag{4.15}$$

We assume just as before that the right-hand sides of equations (4.13,14) make sense, i.e. that $\mathcal{O}\in V_Q$ (cf (7.1) of I).

5. IT within the group algebra

In analogy with I, the first type of IT which we introduce are the elements of the tensor basis of ${}^{(i)}\mathcal{A}((\mathrm{SU}(2)\times\mathrm{SU}(2))$ Ⓢ $\mathrm{S}_2)$. Then we have two possibilities. First they can be defined using the matrix elements of $U^{jj\kappa(j)}$ (see equation (2.12)) and the CG coefficients for (SU(2) × SU(2)) Ⓢ S$_2$:

$$T^{jj\kappa(j);\beta}_p=\sum_{\substack{MK,M'K'\\M''K''}}\left\{\begin{matrix}jj\kappa(j) & jj\kappa(j)\\ MK & M''K''\end{matrix}\middle|\begin{matrix}\beta\\p\end{matrix}\right\}U^{jj\kappa(j)}_{M'K',M''K''}E^{jj\kappa(j)}_{MK,M'K'} \tag{5.1}$$

(cf Kasperkovitz and Dirl 1974). The possible values for $\beta(p)$ for fixed $jj\kappa(j)$ are given by equation (2.30). Therefore $\beta(p)$ can take the following values:

$$\beta=JJ0, \qquad J=0,1,\ldots,2j \qquad (p=MK) \tag{5.2}$$

$$\beta=J_1J_2, \qquad J_i=0,1,\ldots,2j \qquad (p=r'MK). \tag{5.3}$$

The second possibility does not require us to know the CG coefficients for (SU(2) × SU(2)) Ⓢ S$_2$ and therefore allows us to calculate them. For, remembering equations (4.8,9) and the definitions (4.1,2,7,8) of I of the IT within the group algebra of SU(2) × SU(2), it is obvious the two different types of IT components (5.1) are given by:

$$T^{jj\kappa(j);JJ0}_{MK}={}^{(\mathrm{L})}T^{j;J}_M\,{}^{(\mathrm{R})}T^{j;J}_K \tag{5.4}$$

$$T^{jj\kappa(j);J_1J_2}_{eMK}={}^{(\mathrm{L})}T^{j;J_1}_M\,{}^{(\mathrm{R})}T^{j;J_2}_K \tag{5.5}$$

$$T^{jj\kappa(j);J_1J_2}_{sMK}={}^{(\mathrm{L})}T^{j;J_2}_K\,{}^{(\mathrm{R})}T^{j;J_1}_M. \tag{5.6}$$

The relation

$$W(s)^{(L)}T_M^{j;J} = {}^{(R)}T_M^{j;J}W(s) \tag{5.7}$$

being a consequence of equation (3.2) (with $\omega_2 = \omega_0$), is similar to equation (4.7). If we now calculate the matrix elements of the IT components (5.4–6), taking into account that these are multiplicity-free cases, we obtain the following CG coefficients for (SU(2) × SU(2)) Ⓢ S_2:

$$\begin{Bmatrix} JJO & jj\kappa(j) & jj\kappa(j) \\ MK & m'k' & mk \end{Bmatrix} = (JM, jm'|jm)(JK, jk'|jk) \tag{5.8}$$

$$\begin{Bmatrix} J_1J_2 & jj\kappa(j) & jj\kappa(j) \\ eMK & m'k' & mk \end{Bmatrix} = (J_1M, jm'|jm)(J_2K, jk'|jk)\frac{1}{\sqrt{2}} \tag{5.9}$$

$$\begin{Bmatrix} J_1J_2 & jj\kappa(j) & jj\kappa(j) \\ sMK & m'k' & mk \end{Bmatrix} = (J_2K, jm'|jm)(J_1M, jk'|jk)\frac{1}{\sqrt{2}}. \tag{5.10}$$

To obtain this result one has to use equations (3.11), (4.1,2), (2.20,21) of I and (2.37) of this paper. With the aid of equations (2.30,37) we also find the following CG coefficients:

$$\begin{Bmatrix} JJ\kappa & j'j'\kappa' & j''j''\kappa'' \\ MK & m'k' & m''k'' \end{Bmatrix} = (JM, j'm'|j''m'')(JK, j'k'|j''k'')\delta_{\kappa'',\kappa+\kappa'}. \tag{5.11}$$

6. Matrix elements of the unirreps of SU(2) as IT

The second type of IT with respect to (SU(2) × SU(2)) Ⓢ S_2 are just the operators (5.1) of I:

$$[Q_{MK}^{RR}f](\omega) = Q_{MK}^{R}(\omega)f(\omega). \tag{6.1}$$

Because of equation (5.2) of I and

$$W(s)Q_{MK}^{RR}W^{+}(s) = (-1)^{\kappa(R)}Q_{KM}^{RR}, \qquad \kappa(R) = 2R \bmod 2 \tag{6.2}$$

it is obvious that for fixed R the operators (6.1) form an IT of rank $R, R, \kappa(R)$. Using the CG coefficients (5.11) we can rewrite equation (5.4) of I in the following way:

$$Q_{MK}^{RR}Q_{M'K'}^{R'R'} = \sum_{R''}\left(\frac{(2R+1)(2R'+1)}{2R''+1}\right)^{1/2}\begin{Bmatrix} RR\kappa(R) & R'R'\kappa(R') & R''R''\kappa(R'') \\ MK & M'K' & M''K'' \end{Bmatrix}Q_{M''K''}^{R''R''}. \tag{6.3}$$

Therefore the reduced matrix elements are equal to those which are given by the Wigner–Eckart theorem for SU(2) × SU(2).

7. Complete sets of IT composed of Q_{MK}^{RR} and elements of the tensor basis

The special IT with respect to SU(2) × SU(2) which we have introduced in § 6 of I (cf (6.1–8) of I) can be easily extended by means of equations (4.3,6) to IT with respect to (SU(2) × SU(2)) Ⓢ S_2. Of course one has to distinguish between the cases $A = B$ and

$A<B$. In the case $A=B$ we obtain the following equations:

$$W(s)\,{}^{j}T^{(\underline{A}J)AA}_{ab}W^{+}(s)=(-1)^{2A+J}\,{}^{j}T^{(\underline{A}J)AA}_{ba} \tag{7.1}$$

$$W(s)\,{}^{j}T^{(J\underline{A})AA}_{ab}W^{+}(s)=(-1)^{2A+J}\,{}^{j}T^{(J\underline{A})AA}_{ba} \tag{7.2}$$

$$W(s)\,{}^{jj'}T^{(J\underline{R}J')AA}_{ab}W^{+}(s)=(-1)^{2R}\,{}^{jj'}\hat{T}^{(J\underline{R}J')AA}_{ba} \tag{7.3}$$

(To prove these relations one has to use equations (6.1–4,7,8,13,14) of I, (5.7) and (6.2) of this paper). This means that the operators (6.1,2) of I and the sum (difference) of the operators (6.7) and (6.8) of I, i.e.

$${}^{jj'}_{\pm}T^{(J\underline{R}J')AA}_{ab}=\tfrac{1}{2}({}^{jj'}T^{(J\underline{R}J')AA}_{ab}\pm{}^{jj'}\hat{T}^{(J\underline{R}J')AA}_{ab}) \tag{7.4}$$

are already IT of the ranks A, A, κ' with $\kappa'=2A+J,\ 2A+J,\ 2R,\ 2R+1$. In the case $A<B$ we obtain immediately by means of equation (4.6), the following IT:

$$\{{}^{j}T^{(\underline{B}J)AB}_{rab}: r=e,s;\ -A\leqslant a\leqslant A;\ -B\leqslant b\leqslant B\} \tag{7.5}$$

$${}^{j}T^{(\underline{B}J)AB}_{rab}=W(r)\,{}^{j}T^{(\underline{B}J)AB}_{ab}W^{+}(s) \tag{7.6}$$

$${}^{j}T^{(\underline{B}J)AB}_{sab}=(-1)^{A+B-J}\,{}^{j}T^{(\underline{A}J)BA}_{ba} \tag{7.7}$$

$$\{{}^{j}T^{(J\underline{B})AB}_{rab}: r=e,s;\ -A\leqslant a\leqslant A;\ -B\leqslant b\leqslant B\} \tag{7.8}$$

$${}^{j}T^{(J\underline{B})AB}_{rab}=W(r)\,{}^{j}T^{(J\underline{B})AB}_{ab}W^{+}(r) \tag{7.9}$$

$${}^{j}T^{(J\underline{B})AB}_{sab}=(-1)^{A+B-J}\,{}^{j}T^{(J\underline{B})BA}_{ba} \tag{7.10}$$

$$\{{}^{jj'}T^{(J\underline{R}J')AB}_{rab}: r=e,s;-A\leqslant a\leqslant A;-B\leqslant b\leqslant B\} \tag{7.11}$$

$${}^{jj'}T^{(J\underline{R}J')AB}_{rab}=W(r)\,{}^{jj'}T^{(J\underline{R}J')AB}_{ab}W^{+}(r) \tag{7.12}$$

$${}^{jj'}T^{(J\underline{R}J')AB}_{sab}=(-1)^{2R}\,{}^{jj'}\hat{T}^{(J\underline{R}J')BA}_{ba}. \tag{7.13}$$

To prove the relations (7.7,10,13) one has to use once again equations (6.1–4,7,8,13,14) of I, (5.7) and (6.2) of this paper. Now it is obvious that the operators (5.4–6) for $j=j'$ and the operators (7.4,11) for $j\neq j'$ form a complete set of IT provided that (7.3) of I is taken into account. Finally we are able to specify, for the multiplicity-free cases, the CG coefficients for (SU(2)×SU(2)) Ⓢ S_2. For if we compare equation (4.10) with (3.11) of I for a given IT component of the type (7.11) taking into account equation (2.37), we obtain

$$\left\{\begin{matrix} J_1J_2 & j'j'\kappa' \\ eMK & m'k' \end{matrix}\middle|\begin{matrix} jj\kappa \\ mk\end{matrix}\right\}=(J_1M,j'm'|jm)(J_2K,j'k'|jk)\frac{1}{\sqrt{2}} \tag{7.14}$$

$$\left\{\begin{matrix} J_1J_2 & j'j'\kappa' \\ sMK & m'k' \end{matrix}\middle|\begin{matrix} jj\kappa \\ mk\end{matrix}\right\}=(J_2K,j'm'|jm)(J_1M,j'k'|jk)\frac{(-1)^{\kappa+\kappa'}}{\sqrt{2}} \tag{7.15}$$

$$\left\{\begin{matrix} jj\kappa & j'j'\kappa' \\ mk & m'k' \end{matrix}\middle|\begin{matrix} J_1J_2 \\ eMK\end{matrix}\right\}=(jm,j'm'|J_1M)(jk,j'k'|J_2K) \tag{7.16}$$

$$\left\{\begin{matrix} jj\kappa & j'j'\kappa' \\ mk & m'k' \end{matrix}\middle|\begin{matrix} J_1J_2 \\ sMK\end{matrix}\right\}=(jm,j'm'|J_2K)(jk,j'k'|J_1M)(-1)^{\kappa+\kappa'}, \tag{7.17}$$

where the special cases (5.9,10) are contained in equations (7.14,15). (The index w indicating the multiplicity is again omitted.)

Of course it is obvious that the way to construct IT with respect to (SU(2)× SU(2)) Ⓢ S_2 which are composed of the operators Q^{RR}_{MK} and elements of the enveloping algebra is exactly the same as shown before. This gives us a further possibility (namely by means of the operator $W(s)$) to correlate the elements of the left- and right-enveloping algebras, especially the elements of the left- and right-Lie algebras forming together an IT of rank 0, 1 with respect to (SU(2)×SU(2)) Ⓢ S_2. This will be of interest for physical systems whose state spaces are isomorphic to (invariant subspaces or coset spaces of) L^2(SU(2)) and where a symmetry operation of order two (such as the parity for the hydrogen atom (Dirl and Kasperkovitz 1976)) appears.

8. Conclusions

It was the aim of this paper to gain more insight into the problems arising in the construction of IT with respect to a semi-direct product group whose normal subgroup is non-Abelian. Using as basis of our considerations the Hilbert space L^2(SU(2)) as carrier space we discussed the question whether it is possible to trace back any IT (with respect to the semi-direct product group) to IT already known (with respect to the normal subgroup) by means of 'induction'. In so doing we have shown the advantages of constructing IT with respect to the supergroup starting from the IT already known with respect to the normal subgroup. Since the unirreps, the unitary matrices U relating the unirreps to their complex conjugates, the CG coefficients for SU(2) and the unirreps of (SU(2)×SU(2)) Ⓢ S_2 are well known we are able:

(i) to give explicit expressions of IT within the group algebra;

(ii) to construct, without knowing the CG coefficients for (SU(2)×SU(2)) Ⓢ S_2, a complete set of IT (with respect to (SU(2)×SU(2)) Ⓢ S_2) from the IT already known (with respect to SU(2)×SU(2)); and

(iii) to calculate by means of these special IT two classes of CG coefficients for (SU(2)×SU(2)) Ⓢ S_2 which are important when studying selection rules.

These considerations concerning SU(2) can be easily transferred to any finite or compact continuous group G if one knows the unirreps, the unitary matrices U relating the unirreps to their complex conjugates, the CG coefficients for G and the unirreps of $(G\times G)$ Ⓢ S_2. Furthermore this example offers the possibility of calculating some classes of the CG coefficients for the semi-direct product group. Besides this, complete sets of IT and the explicit knowledge of CG coefficients are of interest for all physical systems whose carrier spaces are isomorphic to (invariant subspaces or coset spaces of) $L^2(G)$ and where a finite group of order two plays a major role (as for the hydrogen atom (see Dirl and Kasperkovitz 1976) or for the Bethe–Salpeter equations (see Guth and Soper 1975, Böhm *et al* 1973)). Finally the operator $W(s)$ representing the non-identity element of S_2 offers a further possibility to correlate the elements of the left- and right-enveloping algebras if G is a compact continuous group.

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